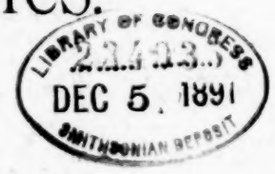
OCTOBER, 1891.

ANNALS OF MATHEMATICS.



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ON CERTAIN SPACE AND SURFACE INTEGRALS.*

By DR. THOMAS S. FISKE, New York, N. Y.

IN the proceedings of the American Academy of Arts and Sciences, May, 1891, Professor B. O. Peirce has given what may be considered as a remarkable extension of a well-known formula of the integral calculus. The formula is †

$$\int\int \frac{du}{dx}\,dxdy = \int u \cos\alpha\, ds\,,$$

the integration in the second member being taken around a closed curve, and that in the first member over the included area, while α denotes the inclination of the exterior normal at any point of the curve to the axis of x. The enunciation of the extension may be given as follows. Let the points of a plane be referred to a system of orthogonal curvilinear co-ordinates $\xi = f_1(x, y)$, $\eta = f_2(x, y)$. If within a closed curve T, the function V and its first space derivatives are finite, continuous, and single-valued, and if the auxiliary quantities h_1, h_2, given by the equations

$$h_1^2 = (D_x\xi)^2 + (D_y\xi)^2,$$
$$h_2^2 = (D_x\eta)^2 + (D_y\eta)^2,$$

are always greater than zero, we have

$$\int\int h_1 h_2 \frac{d}{d\xi}\left[\frac{V}{h_2}\right] d\sigma = \int V \cos\alpha\, ds\,,$$

where the single integration is along the curve T, and the double integration over the included area, while α denotes the inclination of the exterior normal at any point of the curve to the line of constant η at the same point.

Professor Peirce has indicated elsewhere ‡ that a similar result holds in tridimensional co-ordinates.

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* Read before the American Association for the Advancement of Science, August, 1891.

† Byerly's Integral Calculus, 2d edition, p. 196.

‡ Educational Times, February, 1891.

To obtain this, let us suppose that ρ_1, ρ_2, ρ_3 denote orthogonal co-ordinates, that h_1, h_2, h_3 have their usual significance *

$$h_1^2 = \left[\frac{d\rho_1}{dx}\right]^2 + \left[\frac{d\rho_1}{dy}\right]^2 + \left[\frac{d\rho_1}{dz}\right]^2,$$

$$h_2^2 = \left[\frac{d\rho_2}{dx}\right]^2 + \left[\frac{d\rho_2}{dy}\right]^2 + \left[\frac{d\rho_2}{dz}\right]^2,$$

$$h_3^2 = \left[\frac{d\rho_3}{dx}\right]^2 + \left[\frac{d\rho_3}{dy}\right]^2 + \left[\frac{d\rho_3}{dz}\right]^2,$$

and are all greater than zero, and that α is the inclination of the exterior normal at any point of a closed surface T to the line of constant ρ_2 and ρ_3 at the same point. It is to be observed that, when this line passes from without to within the closed surface, α is obtuse, and in the opposite case, acute. We have then

$$\iiint h_1 h_2 h_3 \frac{d}{d\rho_1}\left[\frac{W}{h_2 h_3}\right] dv = \iint W \cos\alpha \, d\sigma .$$

For integrating the first member with respect to ρ_1, we obtain

$$\iiint \frac{d}{d\rho_1}\left[\frac{W}{h_2 h_3}\right] d\rho_1 \, d\rho_2 \, d\rho_3 = \iint \Sigma \pm W \frac{d\rho_2 d\rho_3}{h_2 h_3}$$
$$= \iint W \cos\alpha \, d\sigma ,$$

the last integral following since the infinitely small square $\frac{d\rho_2}{h_2}\frac{d\rho_3}{h_3}$ may be regarded as the projection of $d\sigma$ upon the surface ρ_1. Two other formulas of the same form are obtained by permuting the subscripts.

The results that may be obtained by applying the preceding to polar and cylindrical co-ordinates are of interest. Calculating the auxiliary quantities h by means of the equations of transformation from polar to rectangular co-ordinates, we have

$$h_1 = 1 , \quad h_2 = \frac{1}{r \sin\theta} , \quad h_3 = \frac{1}{r} ;$$

whence

$$\iiint \frac{1}{r^2}\frac{d}{dr}(r^2 W)\, dv = \iint W \cos\alpha \, d\sigma ,$$

$$\iiint \frac{1}{r\sin\theta}\frac{d}{d\varphi} W \, dv = \iint W \cos\beta \, d\sigma ,$$

$$\iiint \frac{1}{r\sin\theta}\frac{d}{d\theta}(\sin\theta . W)\, dv = \iint W \cos\gamma \, d\sigma ;$$

* Todhunter's Functions, p. 220.

α, β, γ being the inclinations of the exterior normal to the lines of constant φ and θ, θ and ρ, ρ and φ, respectively.

In the case of cylindrical co-ordinates

$$h_1 = 1\,, \quad h_2 = \frac{1}{r}\,, \quad h_3 = 1\,.$$

Hence

$$\int\int\int \frac{1}{r}\frac{d}{dr}(rW)\,dv = \int\int W\cos\alpha\,d\sigma\,,$$

$$\int\int\int \frac{1}{r}\frac{d}{d\theta}W\,dv = \int\int W\cos\beta\,d\sigma\,,$$

$$\int\int\int \frac{d}{dz}W\,dv = \int\int W\cos\gamma\,d\sigma\,,$$

α, β, γ being the inclinations of the exterior normal to the lines of constant θ and z, z and r, r and θ, respectively. The last of these formulas is identical with the corresponding one in rectangular co-ordinates.* The preceding two are similar in form to those in plane polar co-ordinates.†

COLUMBIA COLLEGE, *August* 1, 1891.

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*Byerly's Integral Calculus, p. 200.

†Cf. B. O. Peirce, loc. cit.

THE BITANGENTIAL OF THE QUINTIC.

By Mr. Wm. E. Heal, Marion, Indiana.

In a paper published in the Annals for October, 1889, I gave a formula for the bitangential of the quintic, but in an unreduced form. I wish, first, to correct some errors in the numerical coefficients of the equation there given. In the determinant at the bottom of page 40 the factor 5 should be omitted, and in the expression for $J'(u, H, \varphi)$, page 41, the factor 18 was omitted by mistake. Making these changes, the equation of the bitangential becomes

$$|297HJ(u, H, \varphi) + 456HJ'(u, H, \varphi) - 20J(u, H, \theta)|^2 - 3025(4\theta - 9H\varphi)^3 = 0. \quad (1)$$

It was stated at the end of the first paper, above referred to, that "The equation we have found for the bitangential contains an irrelevant factor of the eighteenth order in the variables and of the sixth in the coefficients of the given curve. This factor appears to be the square of the Hessian, and dividing out, the resulting equation is of the forty-eighth order in the variables, and of the eighteenth in the coefficients of the original equation."

It is the purpose of the present paper to show how to transform equation (1) so as to become divisible by H^2.

If we border the matrix of the Hessian both horizontally and vertically with three rows and columns, the resulting determinant is the product, with sign changed, of the two determinants added horizontally and vertically.

Thus, if F, F' be functions of the degrees n, n', we have

$$J(u, H, F)\,J(u, H, F') = \begin{vmatrix} u_1 & u_2 & u_3 \\ H_1 & H_2 & H_3 \\ F_1 & F_2 & F_3 \end{vmatrix} \times \begin{vmatrix} u_1 & u_2 & u_3 \\ H_1 & H_2 & H_3 \\ F_1' & F_2' & F_3' \end{vmatrix}$$

$$= -\begin{vmatrix} a & h & g & u_1 & H_1 & F_1 \\ h & b & f & u_2 & H_2 & F_2 \\ g & f & c & u_3 & H_3 & F_3 \\ u_1 & u_2 & u_3 & & & \\ H_1 & H_2 & H_3 & & & \\ F_1' & F_2' & F_3' & & & \end{vmatrix}, \quad (2)$$

where the subscripts denote differentiation with respect to x, y, z respectively.

This last determinant may be reduced by subtracting from the fourth col-

umn the sum of the first three, multiplied respectively by x, y, z: and likewise with the rows. We thus find

$$J(u,H,F)J(u,H,F') = -\begin{vmatrix} a & h & g & 0 & H_1 & F_1 \\ h & b & f & 0 & H_2 & F_2 \\ g & f & c & 0 & H_3 & F_3 \\ 0 & 0 & 0 & 0 & -9H & -nF \\ H_1 & H_2 & H_3 & -9H & 0 & 0 \\ F_1' & F_2' & F_3' & -n'F' & 0 & 0 \end{vmatrix}$$

$$= nn'FF'\begin{pmatrix} H \\ H \end{pmatrix} - 9nHF\begin{pmatrix} F' \\ H \end{pmatrix} - 9n'HF'\begin{pmatrix} F \\ H \end{pmatrix} + 81H^2\begin{pmatrix} F \\ F' \end{pmatrix}. \quad (3)$$

In particular,

$$[J(u,H,\theta)]^2 = 484\theta^3 - 396H\theta\begin{pmatrix} \theta \\ H \end{pmatrix} + 81H^2\begin{pmatrix} \theta \\ \theta \end{pmatrix}, \quad (4)$$

$$[J(u,H,\varphi)]^2 = 169\theta\varphi^2 - 234H\varphi\begin{pmatrix} \varphi \\ H \end{pmatrix} + 81H^2\begin{pmatrix} \varphi \\ \varphi \end{pmatrix}, \quad (5)$$

$$[J'(u,H,\varphi)]^2 = 36\theta\varphi^2 - 108H\varphi\begin{pmatrix} \varphi' \\ H \end{pmatrix} + 81H^2\begin{pmatrix} \varphi' \\ \varphi' \end{pmatrix}, \quad (6)$$

$$J(u,H,\theta)J(u,H,\varphi) = 286\theta^2\varphi - 117H\varphi\begin{pmatrix} \theta \\ H \end{pmatrix} - 198H\theta\begin{pmatrix} \varphi \\ H \end{pmatrix} + 81H^2\begin{pmatrix} \theta \\ \varphi \end{pmatrix}, \quad (7)$$

$$J(u,H,\theta)J'(u,H,\varphi) = 132\theta^2\varphi - 154H\varphi\begin{pmatrix} \theta \\ H \end{pmatrix} - 198H\theta\begin{pmatrix} \varphi' \\ H \end{pmatrix} + 81H^2\begin{pmatrix} \theta \\ \varphi' \end{pmatrix}, \quad (8)$$

$$J(u,H,\varphi)J'(u,H,\varphi) = 78\theta\varphi^2 - 54H\varphi\begin{pmatrix} \varphi \\ H \end{pmatrix} - 117H\varphi\begin{pmatrix} \varphi' \\ H \end{pmatrix} + 81H^2\begin{pmatrix} \varphi \\ \varphi' \end{pmatrix}; \quad (9)$$

where, after Clebsch, we write

$$\begin{pmatrix} W \\ V \end{pmatrix} = \begin{vmatrix} a & h & g & W_1 \\ h & b & f & W_2 \\ g & f & c & W_3 \\ V_1 & V_2 & V_3 & 0 \end{vmatrix},$$

and the accents denote that in performing the differentiations the second differential coefficients of H, which enter into the expression for φ, are treated as constants. Expanding equation (1) and substituting from equations (4), (5), (6), (7), (8), (9), the equation becomes

$$32400H^2\begin{pmatrix} \theta \\ \theta \end{pmatrix} - 1674675H^2\theta\varphi^2 + 2352240H^2\theta\begin{pmatrix} \varphi \\ H \end{pmatrix}$$

$$-3611520H^2\theta\begin{pmatrix}\varphi'\\H\end{pmatrix}+405000H^2\varphi\begin{pmatrix}\theta\\H\end{pmatrix}+316800H\theta^2\varphi$$
$$-158400H\theta\begin{pmatrix}\theta\\H\end{pmatrix}+2205225H^3\varphi^3-6014250H^3\varphi\begin{pmatrix}\varphi\\H\end{pmatrix}$$
$$+9234000H^3\varphi\begin{pmatrix}\varphi'\\H\end{pmatrix}-962280H^3\begin{pmatrix}\theta\\\varphi\end{pmatrix}+1477440H^3\begin{pmatrix}\theta\\\varphi'\end{pmatrix}$$
$$+7144929H^4\begin{pmatrix}\varphi\\\varphi\end{pmatrix}-21939984H^4\begin{pmatrix}\varphi\\\varphi'\end{pmatrix}+16842816H^4\begin{pmatrix}\varphi'\\\varphi'\end{pmatrix}=0. \qquad (10)$$

Every term is now divisible by H^2 except the two

$$316800H\theta^2\varphi-158400H\theta\begin{pmatrix}\theta\\H\end{pmatrix}=158400H\theta\left[2\theta\varphi-\begin{pmatrix}\theta\\H\end{pmatrix}\right].$$

To reduce this we make use of the following formula :

$$64\left[2\theta\varphi-\begin{pmatrix}\theta\\H\end{pmatrix}\right]=171H^2\psi-72H\begin{pmatrix}\varphi'\\H\end{pmatrix}+\lambda u, \qquad (11)$$

where

$$\psi=\begin{vmatrix}\frac{d^2H}{dx^2} & \frac{d^2H}{dxdy} & \frac{d^2H}{dxdz} & \frac{d}{dx}\\ \frac{d^2H}{dydx} & \frac{d^2H}{dy^2} & \frac{d^2H}{dydz} & \frac{d}{dy}\\ \frac{d^2H}{dzdx} & \frac{d^2H}{dzdy} & \frac{d^2H}{dz^2} & \frac{d}{dz}\\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} & 0\end{vmatrix}u,$$

and λ is a function of x, y, z, which for the purpose of the present paper it is not necessary to determine.

I am not in possession of an *a priori* proof of equation (11); but, as will be seen hereafter, the result to which it leads in the present problem can be verified independently. Substituting equation (11) in equation (10) and dividing by $27H^2$, we have, finally, for the equation of the bitangential,

$$1200\begin{pmatrix}\theta\\\theta\end{pmatrix}-62025\theta\varphi^2+87120\theta\begin{pmatrix}\varphi\\H\end{pmatrix}-140360\theta\begin{pmatrix}\varphi'\\H\end{pmatrix}+15000\varphi\begin{pmatrix}\theta\\H\end{pmatrix}$$
$$+H\begin{bmatrix}81675\varphi^3-15675\theta\psi-35640\begin{pmatrix}\theta\\\varphi\end{pmatrix}\\ +54720\begin{pmatrix}\theta\\\varphi'\end{pmatrix}-222750\varphi\begin{pmatrix}\varphi\\H\end{pmatrix}+342000\varphi\begin{pmatrix}\varphi'\\H\end{pmatrix}\end{bmatrix}$$
$$+H^2\left[264627\begin{pmatrix}\varphi\\\varphi\end{pmatrix}-812592\begin{pmatrix}\varphi\\\varphi'\end{pmatrix}+623808\begin{pmatrix}\varphi'\\\varphi'\end{pmatrix}\right]=0. \qquad (12)$$

Equation (12) can be verified in the same manner as equation (1) was verified in my first paper, and I have thought it worth while to do this for the reason that equation (11) has not been proved. Besides, this verification furnishes a check on the reductions effected in the present paper. It will be sufficient to write $c_1 = c_2 = 0$.

We have then, representing always the sum of the terms of the first and higher order in the function F by T_F,

$$H = b^2c + T_H,$$

$$\theta = 9b^6d_0^2 + 54b^5c^2d_1 + T_\theta,$$

$$\varphi = 6b^4e_0 + 24b^3cd_1 + T_\phi,$$

$$\begin{aligned}\begin{pmatrix}\theta\\ \theta\end{pmatrix} &= 1296b^{14}d_0^2e_0^2 + 7776b^{13}c^2d_0e_0e_1 - 3888b^{13}cd_0^2d_1e_0 + 14256b^{13}cd_0^3e_1\\ &+ 11664b^{12}c^4e_1^2 + 113400b^{12}c^2d_0^2d_1^2 - 36936b^{12}c^3d_0d_2e_0 + 73872b^{12}c^3d_0d_1e_1\\ &+ 42768b^{12}c^3d_0^2e_2 - 99792b^{12}c^2d_0^3d_2 - 110808b^{11}c^3d_2e_1 + 256608b^{11}c^3d_1e_2\\ &- 543348b^{11}c^4d_0d_1d_2 - 138996b^{11}c^4d_0^2d_3 + 662904b^{11}c^4d_1^3\\ &+ 263169b^{10}c^6d_2^2 - 833976b^{10}c^5d_1d_3 + T_{\binom{\theta}{\theta}},\end{aligned}$$

$$\begin{aligned}\begin{pmatrix}\varphi'\\ H\end{pmatrix} &= 108b^7cd_1e_0 - 108b^7cd_0e_1 - 162b^6c^3e_2 - 1674b^6c^2d_0d_2\\ &+ 1728b^6c^2d_1^2 - 1782b^5c^4d_3 + T_{\binom{\phi'}{H}},\end{aligned}$$

$$\begin{aligned}\begin{pmatrix}\varphi\\ H\end{pmatrix} &= 18b^7d_0f_0' + 54b^7c^2f_1' - 144b^7cd_0e_1 + 234b^7cd_1e_0 - 108b^6c^3e_2\\ &- 2142b^6c^2d_0d_2 + 2394b^6c^2d_1^2 - 2214b^5c^4d_3 + T_{\binom{\phi}{H}},\end{aligned}$$

$$\begin{aligned}\begin{pmatrix}\theta\\ H\end{pmatrix} &= 108b^{10}d_0^2e_0 + 432b^9cd_0^2d_1 + 648b^9c^2d_0e_1 + 972b^8c^4e_2\\ &- 3807b^8c^3d_0d_2 + 6075b^8c^3d_1^2 - 3159b^7c^5d_3 + T_{\binom{\theta}{H}},\end{aligned}$$

$$\begin{aligned}\varphi' &= 288b^7d_0e_1 - 288b^7d_1e_0 + 432b^4c^2e_2 - 432b^3c^3d_3\\ &+ 576b^4cd_1^2 - 720b^4cd_0d_2 + T_\psi,\end{aligned}$$

$$\begin{aligned}\begin{pmatrix}\theta\\ \varphi\end{pmatrix} &= 1188b^{11}cd_0^2f_1' + 7128b^{10}c^3d_1f_1' + 216b^{12}d_0e_0f_0' + 648b^{11}c^2e_1f_0'\\ &- 324b^{11}cd_0d_1f_0' - 3078b^{10}c^3d_2f_0' + 1080b^{11}cd_0e_0e_1\\ &- 2376b^{10}c^2d_0^2e_2 + 320076b^{10}cd_0^2d_1^2 - 320076b^{10}cd_0^3d_2 - 5184b^{10}c^3e_1^2\\ &+ 8424b^{10}c^3e_0e_2 - 27864b^{10}c^2d_0d_2e_0 + 13824b^{10}c^2d_0d_1e_1 + 21762b^{10}c^2d_1^2e_0\\ &- 48708b^9c^3d_0^2d_3 + 19440b^9c^4d_1e_2 + 279504b^9c^3d_1^3\\ &- 258768b^9c^3d_0d_1d_2 - 27378b^9c^4d_3e_0 - 401760b^8c^5d_1d_3\\ &+ 116964b^8c^5d_2^2 + T_{\binom{\theta}{\phi}},\end{aligned}$$

$$\begin{Bmatrix} \theta \\ \varphi' \end{Bmatrix} = 3888b^{10}c^3e_0e_2 - 3888b^{10}c^3e_1^2 + 7128b^{10}c^2d_0d_1e_1 - 13608b^{10}c^2d_0d_2e_0$$
$$+ 10044b^{10}c^2d_1^2e_0 - 3564b^{10}c^2d_0^2e_2 + 28512b^{10}cd_0^2d_1^2$$
$$- 28512b^{10}cd_0^3d_2 + 4860b^9c^4d_2e_1 - 12636b^9c^4d_3e_0$$
$$- 5832b^9c^4d_1e_2 - 200556b^9c^3d_0d_1d_2 + 211248b^9c^3d_1^3$$
$$- 39204b^9c^3d_0^2d_3 + 64638b^8c^5d_2^2 - 285768b^8c^5d_1d_3 + T_{\binom{\theta}{\phi'}},$$

$$\begin{Bmatrix} \varphi \\ \varphi \end{Bmatrix} = 36b^{10}f_0^2 - 576b^9ce_1f_0' + 2304b^7c^2e_1^2 - 2736b^7c^2d_2f_0 + 936b^9ce_0f_1$$
$$- 1872b^7c^2e_0e_2 + 3744b^7c^2d_1f_1' + 25272b^7cd_1^2e_0 - 25272b^8cd_0d_2e_0$$
$$+ 21888b^7c^3d_2e_1 - 38376b^7c^3d_3e_0 - 7488b^7c^3d_1e_2$$
$$+ 101088b^7c^2d_1^3 - 101088b^7c^2d_0d_1d_2 + 51984b^6c^4d_2^2$$
$$- 153504b^6c^4d_1d_3 + T_{\binom{\phi}{\phi}},$$

$$\begin{Bmatrix} \varphi \\ \varphi' \end{Bmatrix} = 216b^9ce_0f_1' - 216b^9ce_1f_0 - 756b^7c^2d_2f_0 + 864b^7c^2d_1f_1'$$
$$+ 1728b^7c^2e_1^2 - 1836b^8c^2e_0e_2 + 17064b^7cd_1^2e_0 - 17064b^8cd_0d_2e_0$$
$$- 7344b^7c^3d_1e_2 + 14256b^7c^3d_2e_1 + 68256b^7c^2d_1^3$$
$$- 68256b^7c^2d_0d_1d_2 - 24300b^7c^3d_3e_0 + 28728b^6c^4d_2^2$$
$$- 97200b^6c^4d_1d_3 + T_{\binom{\phi}{\phi'}},$$

$$\begin{Bmatrix} \varphi' \\ \varphi' \end{Bmatrix} = 1296b^8c^2e_1^2 - 1296b^7c^2e_0e_2 + 10368b^7cd_1^2e_0 - 10368b^8cd_0d_2e_0$$
$$- 5184b^7c^3d_1e_2 + 9072b^7c^3d_2e_1 - 14256b^7c^3d_3e_0$$
$$+ 41472b^7c^2d_1^3 - 41472b^7c^2d_0d_1d_2 + 15876b^6c^4d_2^2$$
$$- 57024b^6c^4d_1d_3 + T_{\binom{\phi'}{\phi'}}.$$

The calculation of the absolute term in equation (12) now presents no difficulty, but is extremely tedious.

I found its value to be

$$352836b^{11}(27c^2f_0'^2 + 40d_0^3f_0 + 50ce_0^3 - 25d_0^2e_0^2 - 90cd_0e_0f_0),$$

which, as was shown in the former paper, vanishes if the origin is a point of contact of a bitangent. The correctness of equation (12) is thus established.

SOLUTIONS OF EXERCISES.

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Let four circles, A, B, C, D, pass through a common point, and three straight lines through the same point meet them again in the points A_1, B_1, C_1, D_1; A_2, B_2, C_2, D_2; A_3, B_3, C_3, D_3; then, distinguishing lines in opposite directions by opposite signs, will

$$\begin{vmatrix} A_1B_1 & A_1C_1 & A_1D_1 \\ A_2B_2 & A_2C_2 & A_2D_2 \\ A_3B_3 & A_3C_3 & A_3D_3 \end{vmatrix} = 0.$$

[W. W. *Johnson.*]

SOLUTION.

The equations of the circles may be written

$$x^2 + y^2 + ax + a'y = 0,$$
$$x^2 + y^2 + bx + b'y = 0, \text{ etc.};$$

and those of the lines,

$$y = m_1x, \quad y = m_2x, \quad y = m_3x;$$

whence

$$A_1B_1 = \frac{1}{\sqrt{1 + m_1^2}}[a - b + m_1(a' - b')], \text{ etc.}$$

Putting $a - b = \beta$, $a' - b' = \beta'$, $a - c = \gamma$, etc., and omitting the factors $1/\sqrt{1 - m_1^2}$, $1/\sqrt{1 - m_2^2}$, $1/\sqrt{1 - m_3^2}$, the given determinant becomes

$$\begin{vmatrix} \beta + m_1\beta' & \gamma + m_1\gamma' & \delta + m_1\delta' \\ \beta + m_2\beta' & \gamma + m_2\gamma' & \delta + m_2\delta' \\ \beta + m_3\beta' & \gamma + m_3\gamma' & \delta + m_3\delta' \end{vmatrix} = 0.$$

[*Ormond Stone.*]

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The squares of the sides of a plane triangle added to the squares of the radii of the four circles respectively touching these sides is equal to four times the square of the diameter of the circumscribing circle. [*Yale.*]

SOLUTION.

Let r = radius of inscribed circle,
r_a, r_b, r_c = radii of escribed circles,
a, b, c = the sides of the triangle,
R = radius of circumscribed circle,
$\triangle$ = the area of the triangle.

Then

$$r^2 + r_a^2 + r_b^2 + r_c^2 = \frac{(s-a)(s-b)(s-c)}{s} + \frac{s(s-b)(s-c)}{s-a} + \frac{s(s-a)(s-c)}{s-b} + \frac{s(s-a)(s-b)}{s-c}.$$

Adding $a^2 + b^2 + c^2$ to each side, and substituting for s its value $\frac{1}{2}(a+b+c)$, we may readily obtain for the required sum

$$\frac{a^2b^2c^2}{\triangle^2} = 16R^2.$$

[*T. U. Taylor.*]

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Factor the expression

$$abc(x^3+y^3+z^3) + (a^2b+b^2c+c^2a)(x^2y+y^2z+z^2x) + (a^2c+c^2b+b^2a)(x^2z+z^2y+y^2x) + (a^3+b^3+c^3)xyz + 3abcxyz.$$

[*Yale.*]

SOLUTION.

Expanding we get 27 terms, 9 of which can be factored by xc, 9 by yb, and 9 by zb. Taking the first series we see that 3 can be factored by bx, 3 by cy, and 3 by az. The first series then becomes

$$cx[bx(ax+by+cz) + cy(ax+by+cz) + az(ax+by+cz)],$$

or

$$cx(ax+by+cz)(bx+cy+az).$$

The whole becomes

$$(ax+by+cz)(bx+cy+az)(cx+ay+bz).$$

[*T. U. Taylor.*]

290

Let A, B, C be any three points in space; find the locus of a point whose distances to these three points are in the ratio of three given lines. [*Yale.*]

SOLUTION.

Let A be $(0, 0, 0)$; B, $(a, 0, 0)$; C, $(b, c, 0)$. Then the condition gives

$$\frac{x^2+y^2+z^2}{l^2} = \frac{(x-a)^2+y^2+z^2}{m^2} = \frac{(x-b)^2+(y-c)^2+z^2}{n^2};$$

whence

$$\frac{2ax-a^2}{l^2-m^2} = \frac{2bx+2cy-b^2-c^2}{l^2-n^2},$$

which is of the form $Ax + By + C = 0$, and represents a plane perpendicular to the plane of ABC. [*W. B. Richards.*]

296

(1) A complete plane quadrangle $ABCD$ has six sides AB, AC, . . . which intersect an arbitrary line s in six points P_{AB}, P_{AC}, ... Show (*a*) that a quadrangle may be constructed which shall in this way correspond to *any* five collinear points to which such a notation has been at random but permanently affixed, and (*b*) that all such quadrangles lead to the same sixth point on the line.

(2) A complete quadrilateral $k_1k_2k_3k_4$ has six vertices k_1k_2, k_1k_3, . . . and three diagonals $d_{1234} = (k_1k_2)(k_3k_4)$, ... A transversal s cuts the sides in the points K_1, K_2, K_3, K_4, and the diagonals in the, say, *diagonal* points D_{1234}, D_{1342}, D_{1423}. Show (by a double application of (1, *b*) in each case) that these seven collinear points *with this notation* are *determined*

(*c*) by the four points K and any one of the diagonal points D,

(*d*) by any three of the four points K and any two of the diagonal points D;

i. e. (*ca*) a quadrilateral may be constructed which shall in the way defined correspond to *any* five collinear points to which the notation (*c*) has been at random (but, during the consideration of case (*c*), permanently) affixed, and (*cb*) all such quadrilaterals lead to the same two remaining points; and similarly for case (*d*). [*Yale.*]

SOLUTION.

(1) Let P_{CD} be the sixth point. Through any point A draw the transversals AB, AC, AD intersecting s in P_{AB}, P_{AC}, P_{AD}. Let B be any point in AB. Through B draw BC, BD intersecting s in P_{BC}, P_{BD}. C will be the intersection of AC with BC; similarly, D will be the intersection of AD with BD.

Let E be the intersection of BC and AD. Let $(BCEP_{BC})$, etc. represent the anharmonic ratio of the points designated by the letters within the parentheses. From transversals through BC and s, taking A as the centre of the system, we have $(BCEP_{BC}) = (P_{AB}P_{AC}P_{AD}P_{BC})$, a constant. Taking D as the centre $(BCEP_{BC}) = (P_{BD}P_{CD}P_{AD}P_{BC})$, also necessarily a constant; whence P_{CD} is fixed.

(2) Call the three diagonals d_1, d_2, d_3, and their intersections with s D_1, D_2, D_3. Through any point draw k_1, k_2, d_1 intersecting s in the fixed points K_1, K_2, D_1. Through the fixed point K_3 draw at random k_3. (*c*) Through K_4 and k_3d_1 draw k_4. Through k_1k_3 and k_4k_2 draw d_2. Or, (*d*) through D_2 and k_1k_3 draw d_2. Through k_2d_2 and k_3d_1 draw k_4. In either case, through k_1k_4 and k_2k_3 draw d_3.

From the quadrilateral thus formed, three quadrangles may be formed by omitting, in turn, each of the diagonals and its corresponding vertices. If any five of the points K_1, K_2, K_3, K_4, D_1, D_2 be fixed, (1, b) applied to the corresponding quadrangle fixes the remaining point. If D_3 be substituted for D_1 or D_2 and (1, b) again applied, the seventh point D_3 is fixed.

[*Ormond Stone.*]

310

Solve for $x_1 : x_2 : x_3$ the continued equality

$$(x_2 + x_3)(x_2 - x_3)^2 = (x_3 + x_1)(x_3 - x_1)^2 = (x_1 + x_2)(x_1 - x_2)^2.$$

[*E. H. Moore.*]

SOLUTION.

The geometrical construction of the equations, while yielding the required roots with very little labor of computation, indicates as well the purely algebraic process by which they may be obtained.

Using an equilateral triangle as the triangle of reference, I designate its centre as the point O, and I further name as A, B, and C the vertices of a new triangle formed by drawing through the vertices of the triangle of reference lines parallel to its sides; in other words, these points are the mutual intersections of the lines $x_2 + x_3$, $x_3 + x_1$, and $x_1 + x_2$.

The locus of the equation

$$(x_2 + x_3)(x_2 - x_3)^2 = (x_3 + x_1)(x_3 - x_1)^2, \tag{1}$$

apparently a cubic, is shown to break up into lower curves by expanding each member of the equation and transposing one of them, when we have

$$(x_2^3 - x_1^3) - x_3(x_2^2 - x_1^2) - x_3^2(x_2 - x_1) = 0,$$

or

$$(x_1 - x_2)(x_1^2 + x_1x_2 + x_2^2 - x_1x_3 - x_2x_3 - x_3^2) = 0;$$

and, as the factor in the last parenthesis may be written

$$(x_1 + x_2 + x_3)(x_1 + x_2 - x_3) - (x_2x_3 + x_3x_1 + x_1x_2),$$

it is apparent that the locus consists of the right line $x_1 - x_2$ and a circle.

But in its original form, equation (1) shows that the locus has one of its double points at the intersection of $x_2 - x_3$ with $x_3 - x_1$, that is, at the point O; likewise that $x_3 + x_1$, or AC, is a tangent at the point where it meets $x_2 - x_3$, i. e. at A; and similarly that BC is a tangent at B. From these data the locus is readily constructed, and found to consist of the line OC and the circle AOB. In like manner, the locus of the equation

$$(x_3 + x_1)(x_3 - x_1)^2 = (x_1 + x_2)(x_1 - x_2)^2 \tag{2}$$

is the line OA and the circle BOC.

The nine intersections of these loci are obviously made up as follows: that of OC with OA, viz. O; those of OC with BOC, viz. O and C; those of OA with AOB, viz. O and A; and those of AOB with BOC, which are O and B; together with the imaginary circular points I and J. Hence, as roots of

the two equations, we have, counting as four equal roots the co-ordinates of O, viz. $1 : 1 : 1$; and, each counting singly, those of A, B, and C, viz. $-1 : 1 : 1$, $1 : -1 : 1$, and $1 : 1 : -1$; and finally those of I and J, or $1 : \frac{1}{2}(-1 - \sqrt{-3}) : \frac{1}{2}(-1 + \sqrt{-3})$ and $1 : \frac{1}{2}(-1 + \sqrt{-3}) : \frac{1}{2}(-1 - \sqrt{-3})$.

The number of distinct points of intersection is, of course, not increased by adding for symmetry the locus of the third equation,

$$(x_1 + x_2)(x_1 - x_2)^2 = (x_2 + x_3)(x_2 - x_3)^2. \qquad (3)$$

[*F. H. Loud.*]

311

To find four biquadrate numbers whose sum is a square number.

[*Artemas Martin.*]

SOLUTION.

Let x, $(x - ay)$, $(x - by)$, and $(x - cy)$ represent the numbers required. Then we must have

$$x^4 + (x - ay)^4 + (x - by)^4 + (x - cy)^4 = \square.$$

Expanding and arranging the terms according to the powers of x, we get

$$4x^4 - 4(a + b + c)x^3y + 6(a^2 + b^2 + c^2)x^2y^2 - 4(a^3 + b^3 + c^3)xy^3 + (a^4 + b^4 + c^4)y^4 = \square.$$

Putting this expression

$$= \{2x^2 - (a + b + c)xy + \tfrac{1}{4}[6(a^2 + b^2 + c^2) - (a + b + c)^2]y^2\}^2,$$

expanding, cancelling, and reducing, we find

$$\frac{x}{y} = \frac{[6(a^2 + b^2 + c^2) - (a + b + c)^2]^2 - 16(a^4 + b^4 + c^4)}{8\{(a + b + c)[6(a^2 + b^2 + c^2) - (a + b + c)^2] - 8(a^3 + b^3 + c^3)\}}.$$

[*Artemas Martin.*]

312

Given

$$u = x \sin x + \cos x,$$
$$v = \sin x - x \cos x;$$

to compute

$$\int \frac{x^2 dx}{u^2}, \qquad \int \frac{x^2 dx}{v^2}, \qquad \int \frac{bx^2 dx}{(au + bv)^2}. \qquad [C.\ Hermite.]$$

SOLUTION.

$$(1)\quad \frac{x^2}{u^2} = \frac{\sec^2 x}{\left(\tan x + \frac{1}{x}\right)^2} = \frac{\sec^2 x - \frac{1}{x^2}}{\left(\tan x + \frac{1}{x}\right)^2} + \frac{\frac{1}{x^2}}{\left(\tan x + \frac{1}{x}\right)^2}$$

$$= \frac{\sec^2 x - \frac{1}{x^2}}{\left(\tan x + \frac{1}{x}\right)^2} + \frac{\cot^2 x}{(x + \cot x)^2}.$$

$$\therefore \int \frac{x^2 dx}{u^2} = -\frac{1}{\tan x + \frac{1}{x}} + \frac{1}{x + \cot x} = \frac{-x\cos x + \sin x}{x \sin x + \cos x} = \frac{v}{u}.$$

(2) Similarly, $\int \frac{x^2 dx}{v^2} = -\frac{u}{v}$.

(3) $$\int \frac{bx^2 dx}{(au + bv)^2} = \int \frac{\frac{bx^2}{u^2}\, dx}{\left\{a + b\frac{v}{u}\right\}^2} = -\frac{1}{a + b\frac{v}{u}} = -\frac{u}{au + bv}.$$

[*W. W. Beman.*]

315

A man runs a race starting with velocity v, the direction of the wind appearing to him to make an angle α with his course. It then appears uniformly to veer around through an angle α during the race. The wind blows uniformly throughout and at right angles to his course. Show that if t be the time of the race, the length of the course is $vt\frac{\tan \alpha}{\alpha}\log(2\cos\alpha)$.

[*William Hoover.*]

SOLUTION.

Let x = the distance run at the end of any time t' from the beginning of the race, k = the constant velocity of the wind, and ω = the change per unit of time of the direction of the wind. Then $\alpha + \omega t'$ = the apparent direction of the wind at the time t', and $\frac{dx}{dt'}$ = the man's velocity at the same instant.

$$\therefore \frac{\sin\left\{\alpha + \omega t' - \frac{\pi}{2}\right\}}{\sin(\alpha + \omega t')} = \frac{\frac{dx}{dt'}}{k}. \qquad (1)$$

When $t' = 0$, $dx/dt' = v$; $\therefore -\cot\alpha = v/k$,

whence $k = -v\tan\alpha$, and (1) becomes $dx/dt' = v\tan\alpha\cot(\alpha + \omega t')$. (2)

Multiplying by dt' and integrating, we get $x = \frac{v\tan\alpha}{\omega}\log\sin(\alpha + \omega t') + C$. (3)

When $x = 0$, $t' = 0$; $\therefore C = -\frac{v\tan\alpha}{\omega}\log\sin\alpha$, and (3) is

$$x = v\frac{\tan\alpha}{\omega}\log\frac{\sin(\alpha + \omega t')}{\sin\alpha}. \qquad (4)$$

When $t' = t$, $\omega t' = \omega t = \alpha$, $\omega = \alpha/t$, and the length of the course

$$x_1 = vt\frac{\tan\alpha}{\alpha}\log(2\cos\alpha).$$

[*William Hoover.*]

316

The distance between the heels of a man's feet is $2b$, and the length of each foot is a. As the body sways, the vertical through the centre of gravity must always pass through the area contained by the feet. The toes should therefore be turned out at such an angle that the area contained by the feet is a maximum. Show (1) that a circle can be described about the feet with its centre on the straight line joining the toes, and (2) that the diameter of the circle is $b + (2a^2 + b^2)^{\frac{1}{2}}$. [*E. J. Routh.*]

SOLUTION.

Let $\pi - \theta$ be the angle between the feet. The area contained is

$$\Omega = 2ab \sin\theta + \tfrac{1}{2}a^2 \sin 2\theta.$$

The condition for maximum is

$$d\Omega/d\theta = 2ab\cos\theta + a^2\cos 2\theta = 0,$$

or

$$a \sec\theta = 2b + 2a\cos\theta.$$

Therefore the diagonal of the trapezoid is perpendicular to the sloped side and (1) is true. The diameter of the circle is

$$d = 2b + 2a\cos\theta.$$

Eliminating θ,

$$d = b + \sqrt{(2a^2 + b^2)}.$$ [*W. H. Echols.*]

EXERCISES.

321

A CIRCLE meets a hypocycloid of class 3 at six finite points. Show that the tangents to the hypocycloid at these six points touch a conic.

[*Frank Morley.*]

322 *

THE arc of a limaçon is shown in works on the Calculus to be equivalent to the arc of a certain ellipse. Show that the double point on the limaçon corresponds with Fagnani's point on the ellipse. [*W. B. Richards.*]

323

ONE of two casks contains a gallons of wine, and the other b gallons of water; c gallons are taken from the first and poured into the second cask, and then c gallons are taken from the second and poured into the first. Required the quantity of wine in the second cask after n such operations.

[*Artemas Martin.*]

* Exercise 317 corrected.

324

A SUCTION pump is connected with a reservoir by a pipe of the same diameter as the piston. How many strokes are needed to bring water?

[*W. H. Echols.*]

325

DISCUSS the case of two forces acting together upon a particle, one force being directed to the centre, and the other directed to one focus of an ellipse, under the influence of which forces the particle freely describes the ellipse.

[*Yale Prize Problem.*]

326

A HORIZONTAL beam, span $2a$, is supported at each end; the load per running foot of length at one support is zero; at the other support b. Find the deflection of the beam at the centre due to this load,

(1) When the load increases from zero-support to b-support as the square of the distance;

(2) When the load increases as the square root of that distance.

[*T. U. Taylor.*]

327

FIND an approximate value for the perturbation of a comet by the sun, when the comet is very near a planet. (See Watson's Astronomy, p. 550.)

[*A. Hall.*]

328

A BUBBLE of air is released below the surface of a still pond; discuss its form and motion. [*W. H. Echols.*]

LUDLOW'S TRIGONOMETRY.*

The author of this treatise does not write for mathematicians, but only for those who intend solving triangles, and for their convenience has added trigonometric and logarithmic tables. Regarded in that light, the work is fairly good, but rather lengthy and in some parts inexplicit.

It may be remarked, moreover, that the day has long since departed when students were taught that the sine of an *angle* was the function of the *corresponding arc*. A comparison between this work and those that its author consulted in preparing his own, shows that most of them are to be preferred to the new. Chauvenet, for instance, gives in 16 pages that for which Ludlow requires 39, and in a style considerably to be preferred. Another author gives the same matter (Trigonometric Functions of an Angle) in 12 pages, and in a clearer style than either of them. C. L. D.

* TRIGONOMETRY. By Henry H. Ludlow. 294 pp. 8vo. John Wiley & Sons, New York. 1891.

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ERRATUM.

Page 59, line 3, for

$$= \frac{1}{3\frac{1}{3}} \frac{V}{4g} \int_0^{\sqrt{3}} (\tan^{\frac{1}{3}}\theta + \tan^{\frac{1}{7}}\theta)\, d(\tan\theta) = \frac{44}{153}\frac{1}{}\frac{3}{}\frac{V}{g}.$$

read

$$= \frac{1}{3\frac{1}{3}} \frac{V}{4g} \int_0^{\sqrt{3}} (\tan^{\frac{1}{3}}\theta + \tan^{-\frac{1}{3}}\theta)\, d(\tan\theta) = \frac{4}{3}\frac{V}{g}.$$

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